WINTERSEMESTER 2015/16 - NICHTLINEARE PARTIELLE DIFFERENTIALGLEICHUNGEN

Homework #2 Key

Problem 1. Let $p(x_1, ..., x_N)$ be the $N \times N$ Vandermonde determinant

$$p(x_1, ..., x_N) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{bmatrix} .$$

a.) Show that $p(x_1, ..., x_N)$ is a polynomial of degree $N-1$ in x_N with roots equal $x_1, x_2, ..., x_{N-1}$.

Solution. Expanding the determinant along the last column produces the expression

$p(x_1, .., x_N) = \alpha_{N-1}(x_1, ..., x_{N-1})(x_N^{N-1} + \alpha_{N-2}(x_1, ..., x_{N-1})(x_N^{N-2} + ... + \alpha_0(x_1, ..., x_{N-1})$

which is indeed a polynomial in x_N of degree $N-1$. Furthermore, if $x_N = x_j$ for any $j = 1, 2, .., N-1$, the determinant has two identical columns and must vanish. Hence,

$$p(x_1, ..., x_N) = \beta(x_1, ..., x_{N-1})(x_N - x_1)(x_N - x_2)...(x_N - x_{N-1}) .$$

Note that the same consideration can be made for all the other x_j for $j = 1, 2, ..., N-1$. The determinant p is a polynomial in x_j with zeros $x_1, x_2, ..., x_{j-1}, x_{j+1}, .., x_N$. Thus

$$p(x_1, ..., x_N) = c \prod_{j<k} (x_j - x_k)$$

with c being a non-zero constant.

b.) Show that the $N \times N$ linear system

$$\sum_{j=1}^{N} a_j(-j)^k = 1, \qquad k = 0, 1, ..., N-1$$

has a unique solution.

Solution. Using part a.) the determinant of the coefficient matrix of this system is equal to $p(-1, -2, ..., -N) \neq 0$

Problem 2. Let $s \in (0, 1)$. Show that an equivalent norm in $H^s(\mathbb{R}^d)$ is given by

$$\|u\|_s^2 = \int_{\mathbb{R}^d} |u(x)|^2 dx + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{2s+d}} \, dxdy .$$

Recall that the norm in the Sobolev space $H^s(\mathbb{R}^d)$ is defined using the Fourier transform

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

Proof. Note that after a linear change of variables we have

$$\|u\|_s^2 = \int_{\mathbb{R}^d} |u(x)|^2 dx + \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+z)-u(x)|^2}{|z|^{2s+d}}\, dxdz\ .$$

Using properties of the Fourier transform

$$u(x+z)-u(x) = \mathcal{F}^{-1}[(e^{i\xi z}-1)\hat{u}(\xi)]$$

and hence, using Parseval's identity

$$\int_{\mathbb{R}^d} |u(x+z)-u(x)|^2 = \int_{\mathbb{R}^d} |e^{i\xi z}-1|^2 |\hat{u}(\xi)|^2\, d\xi\ .$$

So far

$$\|u\|_s^2 = \int_{\mathbb{R}^d} \|\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}}^d \left\{ \int_{\mathbb{R}^d} \frac{|e^{i\xi z}-1|^2}{|z|^{2s+d}} dz \right\} |\hat{u}(\xi)|^2\, d\xi\ .$$

In the inner integral, we may assume without loss of generality that $\xi = (|\xi|, 0, ..., 0)$. (Otherwise perform an orthogonal change of coordinates.) The inner integral is then simplified by means of the substitution $y = |\xi| z$

$$\int_{\mathbb{R}^d} \frac{|e^{i|\xi|z_1}-1|^2}{|z|^{2s+d}} dz = |\xi|^{2s} \int_{R^d} \frac{|e^{iy_1}-1|^2}{|y|^{2s+d}}\, dy$$

Since $|e^{iy_1}-1|^2 \le C|y_1|^2$ by means of the Taylor series for the exponential for $|y| \le 1$ small we have

$$\frac{|e^{iy_1}-1|^2}{|y|^{2s+d}} \le C\frac{|y_1|^2}{|y|^{2s+d}} \le C\frac{|y|^2}{|y|^{2s+d}} = C\frac{1}{|y|^{2s+d-2}}$$

for $|y| \le 1$ which shows that the expression is integrable over the unit ball since $s < 1$ (use polar coordinates). So far we have

$$\int_{|y|\le 1} \frac{|e^{iy_1}-1|^2}{|y|^{2s+d}}\, dy = C_1(s)$$

and the integral

$$\int_{|y|\ge 1} \frac{|e^{iy_1}-1|^2}{|y|^{2s+d}}\, dy$$

is convergent because of $s > 0$ (use again polar coordinates). Hence, we have proved that

$$\int_{\mathbb{R}^d} \frac{|e^{iy_1}-1|^2}{|y|^{2s+d}}\, dy = C(s)$$

and thus

$$\|u\|_s^2 = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2\, d\xi + C(s) \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2\, d\xi\ .$$

This norm is equivalent to $\|u\|_{H^s(\mathbb{R}^d)}$ for $0 < s < 1$ because of the inequality

$$c_1|\xi|^{2s} \le [\sqrt{1+|\xi|^2}]^{2s} \le c_2|\xi|^{2s}$$

□

Problem 3. *Sobolev space on the torus* $\mathbb{T}^d$. The torus $\mathbb{T}^d$ is the Cartesian product of d copies of the unit circle S^1. A function defined on the torus is a 2π periodic function with respect to each independent variable. If f is integrable on $\mathbb{T}^d$, then the Fourier coefficients of f are given by

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(\theta) e^{-ik\cdot\theta} d\theta \,, \qquad k \in \mathbb{Z}^d.$$

The set of Fourier coefficients plays for the same role as the Fourier transform in $\mathbb{R}^d$. The expansion of a periodic function into a Fourier series is the analogue of the Fourier inversion formula, that is

$$f(\theta) = \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} \hat{f}(k) e^{ik\cdot\theta}, \qquad \theta \in \mathbb{T}^d \,.$$

One can show that the map $\mathcal{F}$ is a isomorphic between the function spaces $L_2(\mathbb{T}^d)$ and $l_2(\mathbb{Z}^d)$. Then, for $s \in \mathbb{R}$, $s \geq 0$ we define

$$H^s(\mathbb{T}^d) = \left\{ u \in L_2(\mathbb{T}^d) \;:\; \sum_{k\in\mathbb{Z}^d} |\hat{u}(k)|^2 \langle k \rangle^{2s} < \infty \right\}$$

where $\langle k \rangle = \sqrt{1 + |k|^2}$.

a.) Show that $H^s(\mathbb{T}^d)' \approx H^{-s}(\mathbb{T}^d)$ where

$$H^{-s}(\mathbb{T}^d) = \left\{ u \in \mathcal{D}'(\mathbb{T}^d) \;:\; \sum_{k\in\mathbb{Z}^d} |\hat{u}(k)|^2 \langle k \rangle^{-2s} < \infty \right\} \,,$$

Proof. Let $s \geq 0$. Suppose that $u \in H^s(\mathbb{T}^d)$ and let f be a continuous linear functional on $H^s(\mathbb{T}^d)$. Since $H^s(\mathbb{T}^d)$ is a Hilbert space, via the Riesz representation theorem one finds a $v \in H^s(\mathbb{T}^d)$ such that

$$(u, f) = \sum_{k\in\mathbb{Z}^d} \hat{u}(k) \overline{\hat{v}(k)} \langle k \rangle^{2s} \,.$$

This gives $\hat{f}(k) = \hat{v}(k) \langle k \rangle^{2s}$ and since

$$\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2 \langle k \rangle^{-2s} = \sum_{k\in\mathbb{Z}^d} |\hat{v}(k)|^2 \langle k \rangle^{2s} < \infty$$

we have shown $f \in H^{-s}(\mathbb{T}^d)$. So far we have shown that $H^s(\mathbb{T}^d)' \subset H^{-s}(\mathbb{T}^d)$. To prove the converse inclusion let $f \in H^{-s}(\mathbb{T}^d)$. Then for all $u \in H^s(\mathbb{T}^d)$ we have by the Cauchy-Schwarz inequality

$$|(u, f)| = \left| \sum_{k\in\mathbb{Z}^d} \hat{u}(k) \overline{\hat{f}(k)} \right| \leq \left(\sum_{k\in\mathbb{Z}^d} |\hat{u}(k)|^2 \langle k \rangle^{2k} \right)^{1/2} \left(\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2 \langle k \rangle^{-2k} \right)^{1/2} \leq C \|u\|_{H^s(\mathbb{T}^d)} \,,$$

which proves that f defines a bounded linear function on $H^s(\mathbb{T}^d)$. □

b.) Define the operator Λ^σ on $\mathcal{D}'(\mathbb{T}^d)$ by

$$(\Lambda^\sigma u)(\theta) = \sum_{k\in\mathbb{Z}^d} \hat{u}(k) \langle k \rangle^\sigma e^{ik\cdot\theta} \,, \qquad \sigma \in \mathbb{R} \,.$$

Then $H^s(\mathbb{T}^d) = \Lambda^{-s} L_2(\mathbb{T}^d)$ for all $s \in \mathbb{R}$. Show that, for any $s \in \mathbb{R}$, the natural injection operator

$$j : H^{s+\sigma}(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$$

is compact for all $\sigma > 0$. Hint: Note that the mapping $\Lambda^\sigma : H^{s+\sigma}(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$ is continuous and that $j = \Lambda^{-\sigma} \circ \Lambda^\sigma$. Hence, to prove this statement it will suffice to show that $\Lambda^{-\sigma} : H^s(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$ is a compact operator whenever $\sigma > 0$.

Proof. Let $u \in H^s(\mathbb{R}^d)$. Then

$$\sum_{k \in \mathbb{Z}^d} |\hat{u}(k)|^2 \langle k \rangle^{2s} < \infty \, .$$

Introduce the sequence of finite rank operators

$$\Lambda_n^{-\sigma} = \sum_{|k| \le n} \hat{u}(k) \langle k \rangle^{-\sigma} e^{ik \cdot \theta} \, .$$

Here $|k| = |k_1| + |k_2| + ... + |k_d|$. We claim that

$$\lim_{n \to \infty} \|\Lambda^{-\sigma} - \Lambda_n^{-\sigma}\| = 0 \tag{1}$$

where the norm is the operator norm of a linear mapping on $H^s(\mathbb{R}^d)$. Indeed, compute

$$\begin{aligned} \|\Lambda^{-\sigma} - \Lambda_n^{-\sigma}\| &= \sup_{\|u\|_{H^s(\mathbb{T}^d)}=1} \|\Lambda^{-\sigma} u - \Lambda_n^{-\sigma} u\|_{H^s(\mathbb{T}^d)} = \sup_{\|u\|_{H^s(\mathbb{T}^d)}=1} \|\sum_{|k|>n} \hat{u}(k) \langle k \rangle^{-\sigma} e^{ik\cdot\theta}\|_{H^s(\mathbb{T}^d)} \\ &= \sup_{\|u\|_{H^s(\mathbb{T}^d)}=1} \sum_{|k|>n} |\hat{u}(k)|^2 \langle k \rangle^{2s-2\sigma} < \langle n \rangle^{-2\sigma} \sup_{\|u\|_{H^s(\mathbb{T}^d)}=1} \sum_{|k|>n} |\hat{u}(k)|^2 \langle k \rangle^{2s} \\ &= \langle n \rangle^{-2\sigma} \to 0, \qquad n \to \infty \end{aligned}$$

Formula (1) shows that $\Lambda^{-\sigma}$ is the norm limit of finite rank operators. Hence, according to a Theorem from Functional Analysis, $\Lambda^{-\sigma}$ is a compact operator. □